Entanglement

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1. Two Qubits

- q_1, q_2 each one has two possible states $\{|0\rangle, |1\rangle\}$
- 4 possible combinations $|0\rangle \otimes |0\rangle = |00\rangle$
  $|0\rangle \otimes |1\rangle = |01\rangle$
  $\vdots$
- General state $|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$
- More than 2 qubits, n

$$|\psi\rangle = \overbrace{\alpha_{\underbrace{000.0}_{n}}|00..0\rangle + \alpha_{\underbrace{0...01}_{n}}|0..01\rangle \; .... + \alpha_{\underbrace{11..1}_{n}}|1...1\rangle}^{2^n \text{ terms}}$$

2. Measurement on a two Qubits System

- Measurement only gives information about the basis states
- 2 qubits system $\longrightarrow$ 2 bits
- Probability of getting $x \in \{0,1\}^2$ is $|\alpha_x|^2$ $\quad x = 00, 01, 10, 11$
- If we measure $|x\rangle = |ij\rangle$ the new state of q_i is $|i\rangle$ and the new state of q_2 is $|j\rangle$
- what does happen if you only measure 1 qubit?

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

- $Pr(q_1 = 1) = P(q_1 = 1) = P(q_1 = 1, q_0 = 0) + P(q_1 = 1, q_0 = 1) = |\alpha_{10}|^2 + |\alpha_{11}|^2$

- $$|\psi_{new}\rangle = \frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}}$$

3. Entanglement

- Suppose $q_1 \to |\psi_1\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$
  $q_2 \to |\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$q_1 q_2 \to |\psi\rangle = \frac{3}{5\sqrt{2}}|00\rangle + \frac{3}{5\sqrt{2}}|01\rangle + \frac{4}{5\sqrt{2}}|10\rangle + \frac{4}{5\sqrt{2}}|11\rangle$$

- Consider $\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)}$, Can you decompose in $|\psi_1\rangle$ and $|\psi_2\rangle$

$$|\psi_1\rangle = a|0\rangle + b|1\rangle$$

$$|\psi_2\rangle = c|0\rangle + d|1\rangle$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = |\psi\rangle$$

- This is an entangled state $\Longleftrightarrow$ non-separable state
- $$Pr(q_1 = 0) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = Pr(q_1 = 1)$$
- If you measure $q_1 = 0$ what's the new state of the system?

$$|\psi_{new}\rangle = |00\rangle \implies q_1, q_2 = |0\rangle$$

- In entangled states we cannot determine the state of a qubit separately

- Correlation between measurements doesn't depend on the measurement basis $\{|v\rangle, |v^\perp\rangle\}$

$$|0\rangle = \alpha|v\rangle + \beta|v^\perp\rangle \quad , \quad |1\rangle = -\beta|v\rangle + \alpha|v^\perp\rangle$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\Big((\alpha|v\rangle + \beta|v^\perp\rangle) \otimes (\alpha|v\rangle + \beta|v^\perp\rangle)$$
$$+ (-\beta|v\rangle + \alpha|v^\perp\rangle) \otimes (-\beta|v\rangle + \alpha|v^\perp\rangle) \Big)$$
$$= \frac{1}{\sqrt{2}}\left(\cancelto{1}{(\alpha^2+\beta^2)}|vv\rangle + \cancelto{1}{(\alpha^2+\beta^2)}|v^\perp v^\perp\rangle \right)$$
$$= \frac{1}{\sqrt{2}}\left(|vv\rangle + |v^\perp v^\perp\rangle \right)$$

2. Two qubit operators

- Unitary transformations on $\mathbb{C}^4$, 4×4 matrix U, $UU^\dagger = U^\dagger U = I$
- Example

$$CNOT = CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$CNOT|10\rangle = |11\rangle$
$CNOT|00\rangle = |00\rangle$
$CNOT|01\rangle = |01\rangle$
$CNOT|11\rangle = |10\rangle$

Control bit

Controlled bit

- Any unitary transform on two qubits can be closely approximated by sequences of CNOT and single qubit operations
- How to apply a one-qubit operator to one qubit of a two qubits system?